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Generalization of certain well-known polynomial inequalities

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ABSTRACT

If $P(z)=\sum_{\nu=0}^{n}c_\nu z^\nu$ is a polynomial of degree n having no zeros in $|z|<1$, then for $|\beta|\leqslant 1$, it was proved by Jain [V.K. Jain, Generalization of certain well known inequalities for polynomials, Glas. Mat. 32 (52) (1997) 45–51] that

$$\left|zP'(z)+\frac{n\beta}{2}P(z)\right|\leqslant\frac{n}{2}\left\{\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right\}\max_{|z|=1}\left|P(z)\right|,\quad |z|=1.$$

In this paper, we shall first obtain a result concerning minimum modulus of polynomials and next we improve upon the above inequality for the polynomials with restricted zeros. Our results refine and generalize certain well-known polynomial inequalities including some results of Bernstein, Lax, Malik and Vong, and Aziz and Dawood.

1. Introduction and statement of results

If $P(z)$ be a polynomial of degree n, then according to a famous result known as Bernstein's inequality (for reference, see [2]),

$$\max_{|z|=1}\left|P'(z)\right|\leqslant n\max_{|z|=1}\left|P(z)\right|. \tag{1.1}$$

The result is best possible and equality holds for the polynomials having all its zeros at the origin.

If we restrict ourselves to the class of polynomials having no zeros in $|z|<1$, then the inequality (1.1) can be sharpened. In fact, P. Erdös conjectured and later Lax [4] proved that if $P(z)\neq 0$ in $|z|<1$, then (1.1) can be replaced by

$$\max_{|z|=1}\left|P'(z)\right|\leqslant\frac{n}{2}\max_{|z|=1}\left|P(z)\right|. \tag{1.2}$$

The above inequality is sharp and equality holds for $P(z)=\gamma+\delta z^n$, where $|\gamma|=|\delta|=1/2$.

As a generalization of inequality (1.1), it was proved by Jain [3] that if $P(z)$ is a polynomial of degree n, then for $|z|=1$ and $|\beta|\leqslant 1$,

$$\left|zP'(z)+\frac{n\beta}{2}P(z)\right|\leqslant n\left|1+\frac{\beta}{2}\right|\max_{|z|=1}\left|P(z)\right|. \tag{1.3}$$

As an improvement of inequality (1.3) Jain [3] proved that if $P(z)$ has no zeros in $|z|<1$, then for $|\beta|\leqslant 1$ and $|z|=1$,

$$\left|zP'(z)+\frac{n\beta}{2}P(z)\right|\leqslant\frac{n}{2}\left\{\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right\}\max_{|z|=1}\left|P(z)\right|. \tag{1.4}$$

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The result is best possible and equality holds for $P(z) = \gamma + \delta z^n$, where $|\gamma| = |\delta| = 1/2$. For $\beta = 0$ both the inequalities (1.3) and (1.4) reduce to (1.1) and (1.2).

In this paper, we first prove an interesting result concerning minimum modulus of polynomial $P(z)$ and thereby obtain a compact generalization of well-known polynomial inequalities.

Theorem 1. *If $P(z)$ is a polynomial of degree n, having all its zeros in $|z| < 1$, then for every real or complex number β with $|\beta| \leqslant 1$,*

$$\min_{|z|=1}\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \geqslant n\left|1 + \frac{\beta}{2}\right| \min_{|z|=1}\left|P(z)\right|. \tag{1.5}$$

The result is best possible and equality holds for $P(z) = me^{i\gamma}z^n$, $m > 0$.

If we take $\beta = 0$ in Theorem 1, then inequality (1.5) reduces to the following result proved by Aziz and Dawood [1].

Corollary 1. *Let $P(z)$ is a polynomial of degree n, having all its zeros in $|z| < 1$, then*

$$\min_{|z|=1}\left|P'(z)\right| \geqslant n \min_{|z|=1}\left|P(z)\right|.$$

Remark 1. For $\beta = -1$, we get an interesting result from (1.5),

$$\min_{|z|=1}\left|zP'(z) - \frac{n}{2}P(z)\right| \geqslant \frac{n}{2} \min_{|z|=1}\left|P(z)\right|.$$

We next improve upon the inequality (1.4), by using Theorem 1. More precisely, we prove the following

Theorem 2. *If $P(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then for every real or complex number β with $|\beta| \leqslant 1$ and $|z| = 1$,*

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \leqslant \frac{n}{2}\left[\left\{\left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\} \max_{|z|=1}\left|P(z)\right| - \left\{\left|1 + \frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right\} \min_{|z|=1}\left|P(z)\right|\right]. \tag{1.6}$$

Inequality (1.6) is sharp and equality holds for $P(z) = \gamma + \delta z^n$, where $|\gamma| = |\delta| = 1/2$.

For $\beta = 0$, Theorem 2 reduces to the following result proved by Aziz and Dawood [1].

Corollary 2. *Let $P(z)$ is a polynomial of degree n, having no zeros in $|z| < 1$, then for $|z| = 1$*

$$\left|P'(z)\right| \leqslant \frac{n}{2}\left\{\max_{|z|=1}\left|P(z)\right| - \min_{|z|=1}\left|P(z)\right|\right\}.$$

Remark 2. For $\beta = -1$, we get from (1.6),

$$\left|zP'(z) - \frac{n}{2}P(z)\right| \leqslant \frac{n}{2} \max_{|z|=1}\left|P(z)\right| \quad \text{for } |z| = 1,$$

which implies that there is no change in upper bound of $|zP'(z) - \frac{n}{2}P(z)|$ for $|z| = 1$ under the additional condition that $P(z) \neq 0$ in $|z| < 1$ (see [5, inequality (3.9)]).

2. Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 1. *If $P(z)$ is a polynomial of degree n, having all its zeros in the disk $|z| \leqslant 1$, then*

$$\left|zP'(z)\right| \geqslant \frac{n}{2}\left|P(z)\right| \quad \textit{for } |z| = 1. \tag{2.1}$$

Proof. Since all the zeros of $P(z)$ lie in $|z| \leqslant 1$. Hence if $z_1, z_2, \ldots, z_n$ are the zeros of $P(z)$, then $|z_j| \leqslant 1$, $j = 1, 2, \ldots, n$,

$$Re\,\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})} = \sum_{j=1}^{n} Re\,\frac{e^{i\theta}}{e^{i\theta} - z_j} \geqslant \sum_{j=1}^{n}\frac{1}{2} = \frac{n}{2},$$

for every point $e^{i\theta}$, $0 \leqslant \theta < 2\pi$, which is not a zero of $P(z)$. This implies

$$\left|\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})}\right| \geqslant Re\,\frac{e^{i\theta}P'(e^{i\theta})}{P(e^{i\theta})} \geqslant \frac{n}{2},$$

for every point $e^{i\theta}$, $0 \leqslant \theta < 2\pi$. Hence

$$\left|zP'(z)\right| \geqslant \frac{n}{2}\left|P(z)\right| \quad \text{for } |z| = 1,$$

and this completes the proof of Lemma 1. $\square$

Lemma 2. *Let $F(z)$ be a polynomial of degree n, having all its zeros in the disk $|z| \leqslant 1$. If $P(z)$ is a polynomial of degree at most n such that*

$$\left|P(z)\right| \leqslant \left|F(z)\right| \quad \textit{for } |z| = 1,$$

then for $|\beta| \leqslant 1$,

$$\left|z\frac{P'(z)}{n} + \beta\frac{P(z)}{2}\right| \leqslant \left|z\frac{F'(z)}{n} + \beta\frac{F(z)}{2}\right| \quad \textit{for } |z| = 1. \tag{2.2}$$

The above lemma is due to Malik and Vong [5] and the following lemma is due to Jain [3].

Lemma 3. *If $P(z)$ is a polynomial of degree n, then for $|z| = 1$ and $|\beta| \leqslant 1$,*

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| + \left|zQ'(z) + \frac{n\beta}{2}Q(z)\right| \leqslant n\left\{\left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\}\max_{|z|=1}\left|P(z)\right| \tag{2.3}$$

where $Q(z) = z^n\overline{P(1/\bar{z})}$.

3. Proofs of the theorems

Proof of Theorem 1. If $P(z)$ has a zero on $|z| = 1$, then inequality (1.5) is trivial. So we suppose that $P(z)$ has all its zeros in $|z| < 1$. If $m = \min_{|z|=1}|P(z)|$, then $m > 0$ and $|P(z)| \geqslant m$ for $|z| = 1$. Therefore, if λ is a complex number such that $|\lambda| < 1$, then it follows by Rouche's Theorem that the polynomial $G(z) = P(z) - \lambda m z^n$ of degree n, has all its zeros in $|z| < 1$. Applying Lemma 1 to the polynomial $G(z)$, we get

$$\left|zG'(z)\right| \geqslant \frac{n}{2}\left|G(z)\right| \quad \text{for } |z| = 1.$$

Since $G(z)$ has all its zeros in $|z| < 1$, again applying Rouche's Theorem for real or complex number β with $|\beta| \leqslant 1$, it can easily verify that the polynomial $T(z) = zG'(z) + \frac{n\beta}{2}G(z)$ has all its zeros in $|z| < 1$. That is,

$$T(z) = zG'(z) + \frac{n\beta}{2}G(z) \neq 0 \quad \text{for } |z| \geqslant 1.$$

Substituting for $G(z)$, we conclude that for every β, λ with $|\lambda| < 1$, $|\beta| \leqslant 1$ and $|z| \geqslant 1$,

$$T(z) = \left[zP'(z) + \frac{n\beta}{2}P(z)\right] - \lambda\left[mnz^n + \frac{n\beta}{2}mz^n\right] \neq 0. \tag{3.1}$$

This implies for every β with $|\beta| \leqslant 1$ and $|z| \geqslant 1$,

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \geqslant n\left|1 + \frac{\beta}{2}\right|m|z|^n. \tag{3.2}$$

If inequality (3.2) is not true, then there is a point $z = z_0$ with $|z_0| \geqslant 1$ such that

$$\left|z_0P'(z_0) + \frac{n\beta}{2}P(z_0)\right| < n\left|1 + \frac{\beta}{2}\right|m|z_0|^n.$$

We take

$$\lambda = \frac{z_0P'(z_0) + \frac{n\beta}{2}P(z_0)}{n(1 + \frac{\beta}{2})mz_0^n},$$

then $|\lambda| < 1$ and with this choice of λ, we have from (3.1), $T(z_0) = 0$ for $|z_0| \geqslant 1$. But this contradicts the fact that $T(z) \neq 0$ for $|z| \geqslant 1$. Hence in particular, (3.2) gives for every β with $|\beta| \leqslant 1$,

$$\min_{|z|=1}\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \geqslant n\left|1+\frac{\beta}{2}\right|\min_{|z|=1}\left|P(z)\right|.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2. By hypothesis, the polynomial $P(z) \neq 0$ in $|z| < 1$, therefore if $m = \min_{|z|=1}|P(z)|$, then $m \leqslant |P(z)|$ for $|z| \leqslant 1$. Therefore, for a given complex number λ with $|\lambda| \leqslant 1$, then it follows by Rouche's Theorem that the polynomial $G(z) = P(z) - \lambda m$, has no zero in $|z| < 1$. Now if

$$H(z) = z^n\overline{G(1/\bar{z})} = Q(z) - m\bar{\lambda}z^n,$$

then all the zeros of $H(z)$ lie in $|z| < 1$ and $|G(z)| = |F(z)|$ for $|z| = 1$.

Therefore by Lemma 2, we have for $|\beta| \leqslant 1$ and $|z| = 1$,

$$\left|zP'(z) + \frac{n\beta}{2}\left\{P(z) - \lambda m\right\}\right| \leqslant \left|\left\{zQ'(z) - \bar{\lambda}mnz^n\right\} + \frac{n\beta}{2}\left\{Q(z) - \bar{\lambda}mz^n\right\}\right|.$$

This implies

$$\left|\left\{zP'(z) + \frac{n\beta}{2}P(z)\right\} - \frac{n\beta}{2}\lambda m\right| \leqslant \left|\left\{zQ'(z) + \frac{n\beta}{2}Q(z)\right\} - \bar{\lambda}mnz^n\left\{1+\frac{\beta}{2}\right\}\right|. \tag{3.3}$$

Since all the zeros of $Q(z)$ lie in $|z| < 1$, therefore by Theorem 1, we have for $|z| = 1$ and $|\beta| \leqslant 1$

$$\begin{aligned}\left|zQ'(z) + \frac{n\beta}{2}Q(z)\right| &\geqslant n\left|1+\frac{\beta}{2}\right|\min_{|z|=1}\left|Q(z)\right| \\ &= n\left|1+\frac{\beta}{2}\right|m.\end{aligned}$$

Now choosing the argument of λ in (3.3) and letting $|\lambda| \to 1$, we get for $|z| = 1$ and $|\beta| \leqslant 1$,

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| - mn\left|\frac{\beta}{2}\right| \leqslant \left|zQ'(z) + \frac{n\beta}{2}Q(z)\right| - mn\left|1+\frac{\beta}{2}\right|.$$

Equivalently

$$\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \leqslant \left|zQ'(z) + \frac{n\beta}{2}Q(z)\right| - n\left\{\left|1+\frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right\}m.$$

Which implies for every real or complex number β with $|\beta| \leqslant 1$ and $|z| = 1$,

$$2\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \leqslant \left|zP'(z) + \frac{n\beta}{2}P(z)\right| + \left|zQ'(z) + \frac{n\beta}{2}Q(z)\right| - n\left\{\left|1+\frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right\}m.$$

This in conjunction with Lemma 3 gives for $|\beta| \leqslant 1$ and $|z| = 1$,

$$2\left|zP'(z) + \frac{n\beta}{2}P(z)\right| \leqslant n\left[\left\{\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\}\max_{|z|=1}\left|P(z)\right| - \left\{\left|1+\frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right\}m\right]$$

and the theorem follows. □

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